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Generalized Bethe ansatz with the general spin representations of the Sklyanin algebra

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Abstract. The quantum inverse scattering method is applied to the inhomogeneous Heisenberg XYZ model with general spins, i.e. the general spin representations of the Sklyanin algebra. By means of the generalized Bethe ansatz developed by Takhtajan and Faddeev the eigenvectors and eigenvalues of the transfer matrix are found.

1. Introduction

The quantum-mechanical system with the Hamiltonian

$$H = -\frac{1}{2}\sum_{n=1}^{N}(J_x\sigma_n^1\sigma_{n+1}^1 + J_y\sigma_n^2\sigma_{n+1}^2 + J_z\sigma_n^3\sigma_{n+1}^3)$$

is called the Heisenberg XYZ model and has been studied by many physicists and mathematicians because of its importance in the theory of magnetism as well as its fundamental mathematical structure. Here H acts on a Hilbert state space $H^{(N)} = \bigotimes_{m=1}^{N} H_m$, $H_m = \mathbf{C}^2$, J_x, J_y, J_z are real constants and the σ_n^as are the local operators acting on the nth local state space of the chain H_n as the Pauli matrices σ^a:

$$\sigma_n^a = 1_{H_1}\otimes\cdots\otimes 1_{H_{m-1}}\otimes\sigma^a\otimes 1_{H_{m+1}}\otimes\cdots\otimes 1_{H_N}$$

where

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The completely isotropic case $J_x = J_y = J_z$ is called the XXX model, and the eigenvalues and eigenvectors of its Hamiltonian were found by Bethe (1931), who proposed his famous ansatz. This has also been successfully applied to the XXZ model (the case $J_x = J_y \neq J_z$) by Yang and Yang (1966a, b, c).

The completely anisotropic XYZ model requires new technical ideas and was solved first by Baxter (1971a, b, 1972a, b) in his remarkable but almost unaccessibly difficult and technical papers. Baxter made use of the generalization of the Bethe ansatz and the link between the XYZ model and a two-dimensional statistical model

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called the eight vertex model, the transfer matrix of which contains the Hamiltonian of the XYZ model.

The work of Baxter was given a clearer explanation by Takhtajan and Faddeev (1979) from the viewpoint of the quantum inverse scattering method (cf Faddeev (1980) or Kulish and Sklyanin (1982)). Takhtajan and Faddeev, starting from the R-matrix

$$R(u) = \sum_{a=0}^{3} W_a(u)\sigma^a \otimes \sigma^a \tag{1.1}$$

and the L-matrix

$$L(u) = \sum_{a=0}^{3} W_a(u)\sigma^a \otimes \sigma_n^a \qquad (n = 1, \ldots, N) \tag{1.2}$$

described the transfer matrix of the eight vertex model by means of the L-matrix and found the eigenvectors and eigenvalues of the transfer matrix, using a variant of the Bethe ansatz (the 'algebraic Bethe ansatz'†; the original Bethe ansatz is called the 'coordinate Bethe ansatz'; cf Sklyanin (1990) for other variants of the Bethe ansatz). The $W_a(u)$s are determined by the Boltzmann weight of the model. As a result R and L_n satisfy the following important relations:

$$R^{12}(u-v)R^{13}(u)R^{23}(v) = R^{23}(u)R^{13}(v)R^{12}(u-v) \tag{1.3}$$

which is the famous Yang–Baxter relation, and

$$R(u-v)(L_n^1(u)L_n^2(u)) = (L_n^2(u)L_n^1(u))R(u-v). \tag{1.4}$$

The indices of R^{12}, L^1, etc denote, as usual, the tensor components on which the operators act non-trivially.

Since the solvability of the model comes from these relations, there naturally arises the question of generalizating the model by replacement of the L-matrix. This is the motivation of the present work. The operator-valued function $L_n(u)$ which satisfies (1.4) was enumerated by Sklyanin (1982) under the condition that $R(u)$ is the Baxter's R-matrix (1.1) and that $L_n(u)$ has the form

$$L_n(u) = \sum_{a=0}^{3} W_a(u)S_n^a\sigma^a. \tag{1.5}$$

He gave the algebraic relations which characterize S_n^a and constructed a new algebra generated by S_n^a, which is called the 'Sklyanin algebra'. Sklyanin (1983) constructed (probably all) finite-dimensional representations of this algebra, among which are those corresponding to the spin-l representations of sl(2) in a certain limit. The spin-$\frac{1}{2}$ representation of the Sklyanin algebra is $S^a = \sigma^a$, hence the work of Takhtajan and Faddeev (1979) is interpreted as the study of the homogeneous XYZ model with spin-$\frac{1}{2}$.

† Speaking precisely, the algebraic Bethe ansatz for the XYZ model is called the 'generalized' algebraic Bethe ansatz to distinguish it from the XXZ model.

The main goal of the present paper is the generalization of the result of Takhtajan and Faddeev to the inhomogeneous XYZ model with the higher spin in the sense of Sklyanin. That is to say, we shall show that we can replace σ_n^a in (0.2) by the general spin representation matrix of S_n^a, and all the machinery developed by Takhtajan and Faddeev works with little change. This result could be naturally conjectured, when we look at the important role of the algebraic structure (1.1)–(1.4) in the work of Takhtajan and Faddeev.

This paper is organized as follows: in section 2 we review the results from Takhtajan and Faddeev (1979) and Sklyanin (1982, 1983) to fix the notation and normalizations. In section 3 we give the concrete form of the gauge transformation, the local vacuum and the actions of the elements of the modified L-matrices on the local vacuum, and 'do the Bethe ansatz'. In section 4 we add some comments on related topics and possible applications.

We follow the notation from Mumford (1983) for the θ functions. Throughout this paper τ denotes a fixed complex number with a positive imaginary part, and η a fixed complex parameter. Furthermore, we use the following abbreviations:

$$\tau' = -\tau^{-1} \qquad \bar{\tau} = -2\tau^{-1}$$

and for all $a, b = 0, 1$,

$$\theta_{ab}(z) = \theta_{ab}(z;\tau) \qquad \theta'_{ab}(z) = \theta_{ab}(z/\tau;\tau') \qquad \bar{\theta}_{ab}(z) = \theta_{ab}(z/\tau;\bar{\tau}).$$

2. Reviews of the generalized Bethe ansatz and Sklyanin algebra

2.1. The generalized Bethe ansatz for the XYZ model

In this section we briefly review an outline of the generalized Bethe ansatz that will be needed later. First we fix R and L_n as in (1.1) and (1.2), where

$$W_0(\lambda) + W_3(\lambda) = \bar{\theta}_{01}(2\eta)\bar{\theta}_{01}(\lambda)\bar{\theta}_{11}(\lambda+2\eta)$$

$$W_0(\lambda) - W_3(\lambda) = \bar{\theta}_{01}(2\eta)\bar{\theta}_{11}(\lambda)\bar{\theta}_{01}(\lambda+2\eta)$$

$$W_1(\lambda) + W_2(\lambda) = \bar{\theta}_{11}(2\eta)\bar{\theta}_{01}(\lambda)\bar{\theta}_{01}(\lambda+2\eta)$$

$$W_1(\lambda) + W_2(\lambda) = \bar{\theta}_{11}(2\eta)\bar{\theta}_{11}(\lambda)\bar{\theta}_{11}(\lambda+2\eta).$$

Here λ is a complex parameter called the spectral parameter. Written out, L_n is the 2×2 matrix with operator-valued elements:

$$L_n(\lambda) = \begin{pmatrix} W_0\sigma_n^0 + W_3\sigma_n^3 & W_1\sigma_n^1 - \mathrm{i}W_2\sigma_n^2 \\ W_1\sigma_n^1 + \mathrm{i}W_2\sigma_n^2 & W_0\sigma_n^0 - W_3\sigma_n^3 \end{pmatrix} = \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & \delta_n(\lambda) \end{pmatrix}. \tag{2.1}$$

The elements of this matrix are regarded as the operator acting on the Hilbert space $H^{(N)} = \bigotimes_{m=1}^N H_m$, $H_m = \mathbf{C}^2$, trivially on all tensor components but on H_n where σ_n^a act as the Pauli matrices σ^a:

$$\sigma_n^a = 1_{H_1} \otimes \cdots \otimes 1_{H_{n-1}} \otimes \sigma^a \otimes 1_{H_{n+1}} \otimes \cdots \otimes 1_{H_N}.$$

Then (1.3) and (1.4) are satisfied.

The operator matrix

$$T_N(\lambda) := L_N(\lambda)\dots L_2(\lambda)L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{2.2}$$

is called the *monodromy matrix* and the operator on $H^{(N)}$

$$t_N(\lambda) := \operatorname{tr} T_N(\lambda) = A(\lambda) + D(\lambda) \tag{2.3}$$

is called the *transfer matrix* (of the eight vertex model). The ultimate goal is to compute the eigenvalues and eigenvectors of t_N.

We introduce a family of gauge transformations with parameters s and t as follows:

$$\begin{aligned} L_m(\lambda) \mapsto L_m^n(\lambda; s,t) :&= M_{m+1}^n(\lambda; s,t)^{-1} L_m(\lambda) M_m^n(\lambda; s,t) \\ &= \begin{pmatrix} \alpha_m^n & \beta_m^n \\ \gamma_m^n & \delta_m^n \end{pmatrix} \\ T_N(\lambda) \mapsto T_m^n(\lambda; s,t) &= L_N^n \dots L_1^n = {M_{N+1}^n}^{-1} T_N M_1^n \\ &= \begin{pmatrix} A_N^n & B_N^n \\ C_N^n & D_N^n \end{pmatrix} \end{aligned} \tag{2.4}$$

where

$$M_m^n(\lambda; s,t) = \begin{pmatrix} \bar\theta_{11}(s+2(m+n)\eta-\lambda) & c_m^n \bar\theta_{11}(t+2(m+n)\eta+\lambda) \\ \bar\theta_{01}(s+2(m+n)\eta-\lambda) & c_m^n \bar\theta_{01}(t+2(m+n)\eta+\lambda) \end{pmatrix}. \tag{2.5}$$

Here c_m^n is determined so that $\det M_m^n$ is independent of n and m, hence

$$c_m^n = (\bar\theta_{00}((s+t/2)+2(n+m)\eta)\bar\theta_{10}((s+t/2)+2(n+m)\eta))^{-1}.$$

The previously defined $\gamma_m^n(\lambda)$ acts on H_m degenerately and

$$\gamma_m^n(\lambda)\omega_m^n = 0 \tag{2.6}$$

where

$$\omega_m^n = \begin{pmatrix} \bar\theta_{11}(s+2(m+n)\eta) \\ \bar\theta_{01}(s+2(m+n)\eta) \end{pmatrix}. \tag{2.7}$$

Note that ω_m^n is independent of the spectral parameter λ. This is called the *local vacuum*. α_m^n and δ_m^n act on it as follows:

$$\begin{aligned} \alpha_m^n(\lambda)\omega_m^n &= h(2\eta+\lambda)\omega_m^{n-1} \\ \delta_m^n(\lambda)\omega_m^n &= h(\lambda)\omega_m^{n+1} \end{aligned} \tag{2.8}$$

where $h(z) = \bar\theta_{01}(0)\,\bar\theta_{01}(z)\bar\theta_{11}(z)$. Hence A_N^n, C_N^n and D_N^n act on the *generating vectors* defined by $\Omega_N^n := \omega_1^n \otimes \omega_2^n \otimes \dots \otimes \omega_N^n$ as follows:

$$\begin{aligned} A_N^n \Omega_N^n &= h^N(2\eta+\lambda)\Omega_N^{n-1} \\ C_N^n \Omega_N^n &= 0 \\ D_N^n \Omega_N^n &= h^N(\lambda)\Omega_N^{n+1}. \end{aligned} \tag{2.9}$$

More generally we introduce the matrices

$$T_{m,m'}(\lambda; s,t) = M_m^{-1}(\lambda; s,t)T(\lambda)M_{m'}(\lambda; s,t) = \begin{pmatrix} A_{m,m'} & B_{m,m'} \\ C_{m,m'} & D_{m,m'} \end{pmatrix}$$

where

$$M_m(\lambda; s,t) = \begin{pmatrix} \bar{\theta}_{11}(s+2m\eta-\lambda) & c_m^0\bar{\theta}_{11}(t+2m\eta+\lambda) \\ \bar{\theta}_{01}(s+2m\eta-\lambda) & c_m^0\bar{\theta}_{01}(t+2m\eta+\lambda) \end{pmatrix}.$$

Then, clearly, $t(\lambda) = A(\lambda) + D(\lambda) = A_{m,m}(\lambda) + D_{m,m}(\lambda)$ and $T_N^n(\lambda; s,t) = T_{N+n,n}(\lambda; s,t)$.

The commutation relations of $A_{m,m'}$, $B_{m,m'}$, $C_{m,m'}$, $D_{m,m'}$ are derived from

$$R(\lambda-\mu)(T_N(\lambda)\otimes 1)(1\otimes T_N(\mu)) = (1\otimes T_N(\mu))(T_N(\lambda)\otimes 1)R(\lambda-\mu). \quad (2.10)$$

The important relations among them are the following:

$$\begin{aligned} B_{m,m'+1}(\lambda)B_{m+1,m'}(\mu) &= B_{m,m'+1}(\mu)B_{m+1,m'}(\lambda) \\ A_{m,m'}(\lambda)B_{m+1,m'-1}(\mu) &= \alpha(\lambda,\mu)B_{m,m'-2}(\mu)A_{m+1,m'-1}(\lambda) \\ &\quad - \beta_{m'-1}(\lambda,\mu)B_{m,m'-2}(\lambda)A_{m+1,m'-1}(\mu) \\ D_{m,m'}(\lambda)B_{m+1,m'-1}(\mu) &= \alpha(\mu,\lambda)B_{m+2,m'}(\mu)D_{m+1,m'-1}(\lambda) \\ &\quad - \beta_{m+1}(\lambda,\mu)B_{m+2,m'}(\lambda)A_{m+1,m'-1}(\mu) \end{aligned} \quad (2.11)$$

where

$$\alpha(\lambda,\mu) = \frac{h(\lambda-\mu-2\eta)}{h(\lambda-\mu)} \qquad \beta_m(\lambda,\mu) = \frac{h(2\eta)h(\tau_k+\mu-\lambda)}{h(\mu-\lambda)h(\tau_k)}$$

and $\tau_k = (s+t)/2 + 2k\eta - \frac{1}{2}$. Note that these relations are derived only from the form of the R-matrix and not from the structure of the L-matrices.

The generalized Bethe ansatz developed by Takhtajan and Faddeev (1979) is the method of finding the eigenvectors of the transfer matrix $t(\lambda)$ in the form of the linear combination of the vectors

$$\Psi_n(\lambda_1,\ldots,\lambda_M) := B_{n+1,n-1}(\lambda_1)\cdots B_{n+M,n-M}(\lambda_m)\Omega_N^{n-M} \quad (2.12)$$

where $M = N/2$ (N is supposed to be even). The actions of the diagonal elements of the modified monodromy matrix $T_{n,n}(\lambda)$ on this vector is described as follows:

$$\begin{aligned} A_{n,n}(\lambda)\Psi_n(\lambda_1,\ldots,\lambda_M) &= {}_1\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)\Psi_{n-1}(\lambda_1,\ldots,\lambda_M) \\ &\quad + \sum_{j=1}^{M} {}_1\Lambda_j^n(\lambda;\lambda_1,\ldots,\lambda_M)\Psi_{n-1}(\lambda_1,\ldots,\lambda_{j-1},\lambda,\lambda_{j+1},\ldots,\lambda_M) \end{aligned}$$

(2.13)

where

$$
\begin{aligned}
{}_1\Lambda(\lambda;\lambda_1,\ldots,\lambda_M) &= (h(2\eta+\lambda))^N \prod_{k=1}^{M} \alpha(\lambda,\lambda_k) \\
{}_1\Lambda_j^n(\lambda;\lambda_1,\ldots,\lambda_M) &= -\beta_{n-1}(\lambda,\lambda_j)(h(2\eta+\lambda))^N \prod_{k=1,k\neq j}^{M} \alpha(\lambda_j,\lambda_k)
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
D_{n,n}(\lambda)\Psi_n(\lambda_1,\ldots,\lambda_M) &= {}_2\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)\Psi_{n+1}(\lambda_1,\ldots,\lambda_M) \\
&\quad + \sum_{j=1}^{M} {}_2\Lambda_j^n(\lambda;\lambda_1,\ldots,\lambda_M)\Psi_{n+1}(\lambda_1,\ldots,\lambda_{j-1},\lambda,\lambda_{j+1},\ldots,\lambda_M)
\end{aligned}
\tag{2.15}
$$

where

$$
\begin{aligned}
{}_2\Lambda(\lambda;\lambda_1,\ldots,\lambda_M) &= (h(\lambda))^N \prod_{k=1}^{M} \alpha(\lambda_k,\lambda) \\
{}_2\Lambda_j^n(\lambda;\lambda_1,\ldots,\lambda_M) &= -\beta_{n+1}(\lambda,\lambda_j)(h(\lambda))^N \prod_{k=1,k\neq j}^{M} \alpha(\lambda_k,\lambda_j).
\end{aligned}
\tag{2.16}
$$

These are derived from the permutation relations (2.11) and (2.9).

The transfer matrix $t(\lambda) = A_{n,n}(\lambda) + D_{n,n}(\lambda)$ acts on

$$
\Psi_\theta(\lambda_1,\ldots,\lambda_M) = \sum_{n=-\infty}^{\infty} e^{2\pi i n\theta}\Psi_n(\lambda_1,\ldots,\lambda_M)
\tag{2.17}
$$

as

$$
\begin{aligned}
t(\lambda)\Psi_\theta(\lambda_1,\ldots,\lambda_M) &= \{e^{2\pi i\theta}\,{}_1\Lambda(\lambda;\lambda_1,\ldots,\lambda_M) + e^{-2\pi i\theta}\,{}_2\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)\} \\
&\quad \times \Psi_\theta(\lambda_1,\ldots,\lambda_M) + \sum_{n=-\infty}^{\infty}\sum_{j=1}^{M} e^{2\pi i n\theta}\{e^{2\pi i\theta}\,{}_1\Lambda_j^{n+1}(\lambda;\lambda_1,\ldots,\lambda_M) \\
&\quad + e^{-2\pi i\theta}\,{}_2\Lambda_j^{n-1}(\lambda;\lambda_1,\ldots,\lambda_M)\}\Psi_n(\lambda_1,\ldots,\lambda_{j-1},\lambda,\lambda_{j+1},\ldots,\lambda_M).
\end{aligned}
\tag{2.18}
$$

Therefore $\Psi_\theta(\lambda_1,\ldots,\lambda_M)$ is one of the eigenvectors of $t(\lambda)$ with the eigenvalue

$$
e^{2\pi i\theta}\,{}_1\Lambda(\lambda;\lambda_1,\ldots,\lambda_M) + e^{-2\pi i\theta}\,{}_2\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)
\tag{2.19}
$$

provided that λ_j satisfy the *Bethe equations*

$$
\frac{h^N(\lambda_j+2\eta)}{h^N(\lambda_j)} = e^{-4\pi i\theta} \prod_{k=1,k\neq j}^{M} \frac{\alpha(\lambda_k,\lambda_j)}{\alpha(\lambda_j,\lambda_k)}
$$

for all $j = 1,\ldots,M$.

Takhtajan and Faddeev (1979) investigated further the case when 2η is a point of finite order on the elliptic curve with the period $(1,\tau)$. We do not repeat their results here.

2.2. *Sklyanin algebra*

In this section we review the algebra constructed by Sklyanin (1982) and its representations studied by Sklyanin (1983).

Define $R(\lambda)$ by (1.1) with $W_a(\lambda)$ as in section 2. Then the necessary and sufficient condition for $L(\lambda) = L_n(\lambda)$ with the form (1.5) to be the solution of (1.4) is that $S^a = S^a_n$ satisfies the following commutation relations (we omit the indices n in this section):

$$\begin{aligned} [S^\alpha, S^0]_- &= -\mathrm{i} J_{\alpha,\beta}[S^\beta, S^\gamma]_+ \\ [S^\alpha, S^\beta]_- &= \mathrm{i}[S^0, S^\gamma]_+ \end{aligned} \tag{2.20}$$

where (α, β, γ) stands for any cyclic permutation of (1, 2, 3), $[A, B]_\pm = AB \pm BA$, and $J_{\alpha,\beta} = (W_\alpha^2 - W_\beta^2)/(W_\gamma^2 - W_0^2)$, i.e.

$$J_{12} = \frac{\theta_{01}(\eta)^2\theta_{11}(\eta)^2}{\theta_{00}(\eta)^2\theta_{10}(\eta)^2} \qquad J_{23} = \frac{\theta_{10}(\eta)^2\theta_{11}(\eta)^2}{\theta_{00}(\eta)^2\theta_{01}(\eta)^2} \qquad J_{31} = -\frac{\theta_{00}(\eta)^2\theta_{11}(\eta)^2}{\theta_{01}(\eta)^2\theta_{10}(\eta)^2}. \tag{2.21}$$

The algebra Q generated by $S^0, \ldots, S^3$ following the relations (2.20) is called the *Sklyanin algebra* (cf also Odesskiĭ and Feĭgin (1989)). Sklyanin (1983) studied the representations of this algebra extensively. We use in this paper the following 'spin-l' representations (l is a half integer):

$$\rho_l : Q \to \mathrm{End}(V_l).$$

The representation space is the subspace of the space of entire functions on the complex plane defined by

$$V_l := \Theta_{00}^{4l+} = \{f(v) | f(v+1) = f(-v) = f(v), f(v+\tau) = \mathrm{e}^{-4l\pi\mathrm{i}(2v+\tau)} f(v)\}. \tag{2.22}$$

It is easy to see that $\dim V_l = 2l + 1$.

The generators of the algebra act on this space as follows:

$$(S^a f)(v) = \frac{s_a(v - l\eta) f(v+\eta) - s_a(-v - l\eta) f(v - \eta)}{\theta_{11}(2v)} \tag{2.23}$$

where

$$\begin{aligned} s_0(v) &= \theta_{11}(\eta)\theta_{11}(2v) \qquad & s_1(v) &= \theta_{10}(\eta)\theta_{10}(2v) \\ s_2(v) &= \mathrm{i}\theta_{00}(\eta)\theta_{00}(2v) \qquad & s_3(v) &= \theta_{01}(\eta)\theta_{01}(2v). \end{aligned}$$

These representations reduce to the usual spin-l representations of U(sl(2,$\mathbb{C}$)) for $J_{\alpha\beta} \to 0$ ($\eta \to 0$). In particular, in the case $l = \frac{1}{2}$, S^a are expressed by the Pauli matrices σ^a as follows: Take $(\theta_{00}(2v; 2\tau) - \theta_{10}(2v; 2\tau), \theta_{00}(2v; 2\tau) + \theta_{10}(2v; 2\tau))$ as the basis of $V_{1/2} = \Theta_{00}^{2+}$. With respect to this basis S^a are written in the matrix form

$$S^a = 2\frac{\theta_{00}(\eta)\theta_{01}(\eta)\theta_{10}(\eta)\theta_{11}(\eta)}{\theta_{00}(0)\theta_{01}(0)\theta_{10}(0)}\sigma^a.$$

Since the relations (2.20) are homogeneous, the overall constant factor in the representation is not essential.

3. Main result

In this section we generalize the result of Takhatajan and Faddeev (1979) by means of the Sklyanin algebra and its representations. By the gauge transformation, which is essentially the same as that of Takhtahan and Faddeev, we can construct eigenvectors of the transfer matrix of the inhomogeneous lattice with general spins.

The total Hilbert space considered now is

$$H^{(N)} = \bigotimes_{m=1}^{N} H_m$$

where the local state space H_m is the spin-l_m representation space of the Sklyanin algebra $V_{l_m}(=\mathbb{C}^{2l_m+1})$, $l_m \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$.

The L-matrix acting on the mth site is

$$L_m(\lambda; l_m) = \sum_{a=0}^{3} W_a(\lambda)\rho_{l_m}(S^a) \otimes \sigma_m^a \tag{3.1}$$

where ρ_l is the spin-l representation of the Sklyanin algebra defined in section 2.2, and S^a are the generators of this algebra. Here in this section we use for the convenience of the computations

$$W_0(\lambda) = \frac{\theta_{11}(\eta+\lambda)}{\theta_{11}(\eta)} \qquad W_1(\lambda) = \frac{\theta_{10}(\eta+\lambda)}{\theta_{10}(\eta)}$$
$$W_2(\lambda) = \frac{\theta_{00}(\eta+\lambda)}{\theta_{00}(\eta)} \qquad W_3(\lambda) = \frac{\theta_{01}(\eta+\lambda)}{\theta_{01}(\eta)}$$

which is proportional to those used in section 2.

We define the inhomogeneous monodromy matrix $T(\lambda, \lambda^{(0)})$ and the transfer matrix $t(\lambda)$ as follows:

$$\begin{aligned} T(\lambda; \lambda^{(0)}) &:= L_N(\lambda - \lambda^{(0)}{}_N; l_N) \cdots L_1(\lambda - \lambda^{(0)}{}_1; l_1) \\ t(\lambda; \lambda^{(0)}) &:= \operatorname{tr} T(\lambda; \lambda^{(0)}). \end{aligned} \tag{3.2}$$

Here $\lambda^{(0)}{}_m$ are the fixed parameters. As in section 2.1 the goal is to construct eigenvectors $\in H^{(N)}$ of $t(\lambda; \lambda^{(0)})$.

First we introduce the gauge transformation of the L-matrix. In order to apply the generalized Bethe ansatz to our case, we need a family of the gauge transformations

$$L_m(\lambda; l_m) \mapsto \tilde{L}_m^n(\lambda; l_m) = \begin{pmatrix} \alpha_m^n(\lambda) & \beta_m^n(\lambda) \\ \gamma_m^n(\lambda) & \delta_m^n(\lambda) \end{pmatrix}$$

with the corresponding local vacuum $\omega_m^n \in H_m$, such that ω is independent of λ and

$$\gamma_m^n(\lambda)\omega_m^n = 0$$

and $\alpha_m^n(\lambda)\omega_m^n$, $\delta_m^n(\lambda)\omega_m^n$ have simple forms. The desired transformation is almost the same as that for the spin-$\frac{1}{2}$ case:

$$L_m(\lambda; l_m) \mapsto \tilde{L}_m^n(\lambda; s, t; l_m) = M_m^n(\lambda; s, t; l_m) L_m(\lambda; l_m) M_{m-1}^n(\lambda; s, t; l_m)^{-1} \tag{3.4}$$

where

$$M_m^n(\lambda; s, t; l_m) = M_{2l_m m+n}(\lambda; s, t). \tag{3.5}$$

$M_k(\lambda; s, t)$ is defined in section 2.1 after (2.9). The local vacuum $\omega_m^n(s)$ is defined up to the constant factor:

$$\omega_m^n(s; v) = a_m^n(s)\,\tilde{\omega}_m^n(s; v) \in \Theta_{00}^{4l+} \tag{3.6}$$

where

$$\tilde{\omega}_m^n(s; v) = \prod_{k=1}^{2l_m} \theta\left(v + \frac{s}{2} + \frac{\tau}{4} + (2l_m m + n)\eta - (3l_m + \tfrac{1}{2})\eta + 2k\eta\right)$$
$$\times\, \theta\left(v - \frac{s}{2} - \frac{\tau}{4} - (2l_m m + n)\eta + (3l_m + \tfrac{1}{2})\eta - 2k\eta\right)$$

and

$$\frac{a_{m,n-1}}{a_{m,n}} = \exp\left\{-\pi \mathrm{i}\frac{2l_m\eta}{\tau}\left[4\left(\frac{s}{2} + (2l_m m + n)\eta - l_m\eta\right) + \tau\right]\right\}.$$

$\gamma_m^n(\lambda)$, $\alpha_m^n(\lambda)$, $\delta_m^n(\lambda)$ act on $\{\omega_m^n(s)\}_{n\in\mathbb{Z}}$ as follows:

$$\begin{aligned}
\gamma_m^n(\lambda)\omega_m^n(s) &= 0 \\
\alpha_m^n(\lambda)\omega_m^n(s) &= h_+^{(l_m)}(\eta, \lambda)\omega_m^{n-1}(s) \\
\delta_m^n(\lambda)\omega_m^n(s) &= h_-^{(l_m)}(\eta, \lambda)\omega_m^{n+1}(s)
\end{aligned} \tag{3.7}$$

where

$$h_\pm^{(l_m)}(\eta, \lambda) = \pm 2\exp\left(\frac{4\pi \mathrm{i} l_m\eta}{\tau}[\pm(\lambda + \eta) - \eta]\right)\theta_{11}(2l_m\eta \pm (\lambda + \eta)). \tag{3.8}$$

The apparent discrepancy between (3.7) and (2.8) comes from the difference between the normalizations of $W_a(\lambda)$ and $\rho_l(S^a)$.

The verification of (3.7) is essentially straightforward but is a terribly tedious computation using the Riemann's relations, the Landen transformations and the modular transformations (cf Mumford (1983) or Whittaker and Watson (1927)). As an example, we sketch the strategy of the calculation of $\gamma_m^n(\lambda)\omega_m^n(s)$.

$\gamma_m^n(\lambda)$ acts on $f(v) \in V_l = \Theta_{00}^{4l+}$ $(v \in \mathbf{C})$ as

$$(\gamma_m^n(\lambda)f)(v) = \frac{C}{\theta_{11}(2v)}(\Gamma_+(v)f(v + \eta) - \Gamma_-(v)f(v - \eta))$$

where C is a constant independent of v, and

$$\Gamma_+(v) = \theta\left(v - \frac{\tau}{4} - \frac{s}{2} - (2l_m m + n - l_m - \tfrac{1}{2})\eta + \lambda\right)$$
$$\times\, \theta\left(v + \frac{\tau}{4} + \frac{s}{2} + (2l_m m + n - l_m - \tfrac{1}{2})\eta - \lambda\right)$$
$$\times\, \theta\left(v + \frac{\tau}{4} + \frac{s}{2} + (2l_m m + n - 3l_m + \tfrac{1}{2})\eta\right)$$
$$\times\, \theta\left(v - \frac{\tau}{4} - \frac{s}{2} - (2l_m m + n + l_m + \tfrac{1}{2})\eta\right)$$

$$\Gamma_-(v) = \Gamma_+(-v).$$

Putting $f(v) = \omega_m^n(s;v)$, we can prove the first equation of (3.7). We omit the detail.

Now we turn to the study of the monodromy matrix and the transfer matrix. We fix the parameters s, t and n and set

$$s_m := s - \lambda^{(0)}{}_m \qquad t_m := t + \lambda^{(0)}{}_m \qquad n_m := n + 2\sum_{k=1}^{m-1}(l_k - l_m). \tag{3.9}$$

Noting that

$$\begin{aligned} M^{n_{m+1}}_{(m+1)-1}(\lambda - \lambda^{(0)}{}_{m+1}; s_{m+1}, t_{m+1}; l_{m+1}) &= M^{n_m}_m(\lambda - \lambda^{(0)}{}_m; s_m, t_m; l_m) \\ &= M_{2l_m m + n_m}(\lambda; s, t) \end{aligned} \tag{3.10}$$

we can transform the monodromy matrix in a simple way by means of the gauge transformation introduced earlier:

$$\begin{aligned} T_N(\lambda;\lambda^{(0)}) &\mapsto T^n_N(\lambda; s, t; \lambda^{(0)}) \\ &:= \tilde{L}^{n_N}_N(\lambda - \lambda^{(0)}{}_N; s_N, t_N; l_N) \cdots \tilde{L}^{n_1}_1(\lambda - \lambda^{(0)}{}_1; s_1, t_1; l_1) \\ &= M^{-1}_{2l_N N + n_N}(\lambda; s, t) T_N(\lambda; \lambda^{(0)}) M_{n_1}(\lambda; s, t). \end{aligned} \tag{3.11}$$

We denote the elements of $T^n_N(\lambda;\lambda^{(0)})$ by

$$T^n_N(\lambda;\lambda^{(0)}) = \begin{pmatrix} A^n_N(\lambda) & B^n_N(\lambda) \\ C^n_N(\lambda) & D^n_N(\lambda) \end{pmatrix}.$$

Then the action of A^n_N, C^n_N, D^n_N act on the *generating vectors* in $H^{(N)}$

$$\Omega^n_N(s;\lambda^{(0)}) = \omega_1^{n_1}(s_1) \otimes \cdots \otimes \omega_N^{n_N}(s_N) \qquad n \in \mathbb{Z}$$

as follows.

$$\begin{aligned} C^n_N(\lambda)\Omega^n_N(s;\lambda^{(0)}) &= 0 \\ A^n_N(\lambda)\Omega^n_N(s;\lambda^{(0)}) &= h_+^{(l_1,\ldots,l_N)}(\eta,\lambda)\Omega^{n-1}_N(s;\lambda^{(0)}) \\ D^n_N(\lambda)\Omega^n_N(s;\lambda^{(0)}) &= h_-^{(l_1,\ldots,l_N)}(\eta,\lambda)\Omega^{n+1}_N(s;\lambda^{(0)}) \end{aligned} \tag{3.12}$$

where

$$h_\pm^{(l_1,\ldots,l_N)}(\eta,\lambda) = \prod_{k=1}^{N} h_\pm^{(l_j)}(\eta, \lambda - \lambda^{(0)}{}_j). \tag{3.13}$$

As in section 2.1 we define more general transformations of T_N by

$$T_N(\lambda;\lambda^{(0)}) \mapsto T_{m,m'}(\lambda; s, t; \lambda^{(0)}) := M^{-1}_m(\lambda; s, t) T_N(\lambda; \lambda^{(0)}) M_{m'}(\lambda; s, t). \tag{3.14}$$

So

$$T_N^n(\lambda; s, t; \lambda^{(0)}) = T_{2l_N N + n_N, n_1}(\lambda; s, t; \lambda^{(0)}). \tag{3.15}$$

The commutation relations of $A_{m,m'}$, $B_{m,m'}$, $D_{m,m'}$, defined by

$$T_{m,m'}(\lambda) = \begin{pmatrix} A_{m,m'}(\lambda) & B_{m,m'}(\lambda) \\ C_{m,m'}(\lambda) & D_{m,m'}(\lambda) \end{pmatrix} \tag{3.16}$$

are, as in section 2.1, derived from the fundamental relation

$$\begin{aligned} &R(\lambda - \mu)(T_N(\lambda; \lambda^{(0)}) \otimes 1)(1 \otimes T_N(\mu; \lambda^{(0)})) \\ &\quad = (1 \otimes T_N(\mu; \lambda^{(0)}))(T_N(\lambda; \lambda^{(0)}) \otimes 1) R(\lambda - \mu) \end{aligned} \tag{3.17}$$

which is the direct consequence of (1.4). Since we use the same (or, exactly speaking, proportional) R-matrix as in section 2.1 and the commutation relations depend only on the form of the R-matrix, as emphasized in section 2.1, (2.11) holds in the present case without changes, so we do not rewrite them.

On the basis of these data, we can follow the method of the generalized Bethe ansatz. Keeping in mind (3.15) and the fact

$$(2l_N N + n_N) - n_1 = 2l_{\text{total}} \qquad l_{\text{total}} := \sum_{k=1}^{N} l_k \tag{3.18}$$

we define $\Psi_n(\lambda_1, \ldots, \lambda_M)$ by (2.12), where $M = l_{\text{total}}$, provided that l_{total} is an integer. Equations (2.13)–(2.16) hold, if we replace the factor $(h(2\eta + \lambda))^N$ and $(h(\lambda))^N$ by $h_+^{(l_1,\ldots,l_N)}(\eta, \lambda)$ and $h_-^{(l_1,\ldots,l_N)}(\eta, \lambda)$ respectively. Therefore Ψ_θ defined by (2.17) is one of the eigenvectors of $t(\lambda; \lambda^{(0)})$ with the eigenvalue

$$\mathrm{e}^{2\pi\mathrm{i}\theta}\,{}_1\Lambda(\lambda; \lambda_1, \ldots, \lambda_M) + \mathrm{e}^{-2\pi\mathrm{i}\theta}\,{}_2\Lambda(\lambda; \lambda_1, \ldots, \lambda_M) \tag{3.19}$$

if λ_j satisfy the Bethe equations

$$\frac{h_+^{(l_1,\ldots,l_N)}(\eta, \lambda)}{h_-^{(l_1,\ldots,l_N)}(\eta, \lambda)} = \mathrm{e}^{-4\pi\mathrm{i}\theta} \prod_{k=1, k\neq j}^{M} \frac{\alpha(\lambda_k, \lambda_j)}{\alpha(\lambda_j, \lambda_k)}$$

for all $j = 1, \ldots, M$.

The rest of the results from Takhtajan and Faddeev (1979), concerning the case when 2η is a point of finite order of the elliptic curve, can also be generalized to the general spin inhomogeneous chain. But it is only a literal translation, so we leave it to the reader.

4. Concluding remarks

Here we make some additional remarks on related topics and possible further applications.

(i) As briefly commented on by Takhtajan and Faddeev (1979), the (low) excitation spectrum of the inhomogeneous higher spin XYZ model can be calculated by means of the integral equation method. Reshetikhin (1990) showed recently that the excitation spectrum for the spin $> \frac{1}{2}$ differs significantly from the case spin $= \frac{1}{2}$ even for the XXX antiferromagnet model in that respect that it has an internal degree of freedom described in terms of some RSOS model. Our results could perhaps help to generalize Reshetikhin's results to the XYZ case and to see what RSOS model will then appear.

(ii) There are several results connected to the Sklyanin algebra in the theory of RSOS lattice models (for example, Hasegawa and Yamada (1990), Hasegawa (1990)). There might be some relation between those results and our work.

(iii) The elliptic R-matrices of higher rank are called the Belavin's R-matrix (cf Vershik (1984) and Cherednik (1986)). It is expected that our strategy will work for these R-matrices and that the generalized Bethe ansatz will give eigenvalues and eigenvectors.

(iv) As the study of the trigonometric R-matrices and the quantum inverse scattering method associated with it lead to the study of the quantum groups (cf Drinfeld (1986)), the study of the elliptic R-matrix and the physical systems associated with it will contribute to the understanding of the Sklyanin algebra (or similar algebras) and the representation theory. To this end our results together with the results quoted in (ii) and the functional Bethe ansatz of the XYZ model (cf Sklyanin (1985a, b, 1986)) could play an important role.

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